Closing Tues: 13.2, 13.3
Closing Thur: 13.4
Exam 1 is Thurs (April 19)
covers 12.1-12.6, 13.1-13.4

Entry Task:

$$\vec{\boldsymbol{r}}(t) = \langle 2\cos(t), 2\sin(t), 0 \rangle$$

Find $\vec{\boldsymbol{T}}(t)$, $\vec{\boldsymbol{N}}(t)$, and K.

13.1-13.4 Curves in 3D

Given $\vec{\boldsymbol{r}}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{\boldsymbol{r}}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ tangent vector

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(u)| du = \text{distance}$$

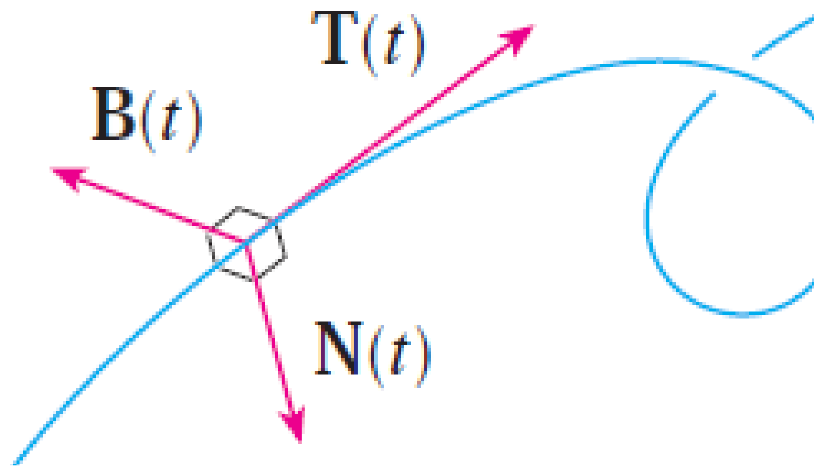
$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3} = \text{curvature}$$

Note:

$\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.



Proof:

Since $\boldsymbol{T} \cdot \boldsymbol{T} = |\boldsymbol{T}|^2 = 1$, we can differentiate both sides to get

$$\boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}' = 0.$$

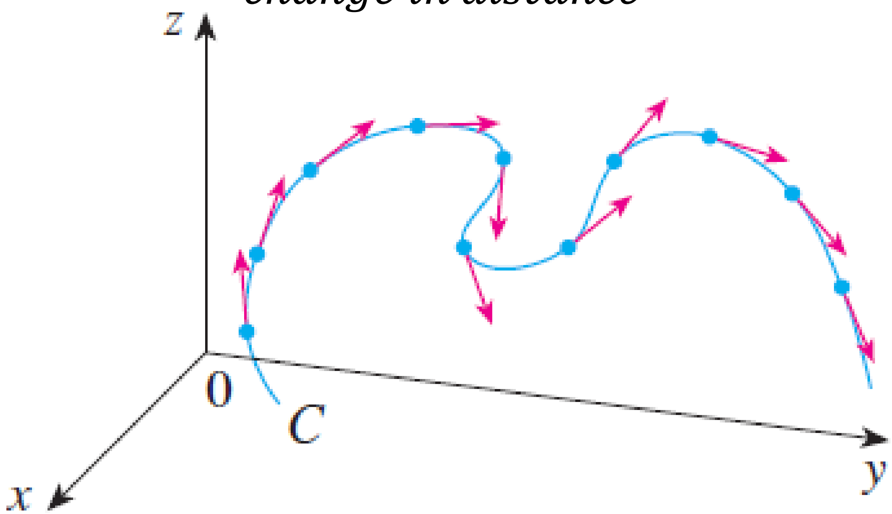
So $2\boldsymbol{T} \cdot \boldsymbol{T}' = 0$.

Thus, $\boldsymbol{T} \cdot \boldsymbol{T}' = 0$. (QED)

Curvature

The **curvature** at a point, *K*, is a measure of how quickly a curve is changing direction at that point.

$$K = \frac{\text{change in direction}}{\text{change in distance}}$$



Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left|\frac{\vec{T_2} - \vec{T_1}}{\text{"one inch"}}\right| = \left|\frac{\Delta\vec{T}}{\Delta s}\right|$$

So we define:

$$K = \left|\frac{d\vec{T}}{ds}\right|.$$

Computation Notes

(see my 13.3 Notes/book for the proof)

1st shortcut:

$$K(t) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

2nd shortcut

$$K(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Aside:

The *radius of curvature* is the radius of the circle that would best fit the curve at the given point.

$$\textit{radius of curvature} = \frac{1}{K}$$

2D Curvature

To find curvature for, $y = f(x)$,
We form the 3D vector function

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle.$$

so $\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Example: $f(t) = x^2$
At what point (x, y, z) is the curvature maximum?

13.4 Position, Velocity, Acceleration

If *t = time* and position is given by

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$

then

$$\boldsymbol{r}'(t) = \lim_{h\to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

$$= \frac{\text{change in position}}{\text{change in time}}$$

$$= \text{velocity} = \boldsymbol{v}(t)$$

$$|\boldsymbol{r}'(\boldsymbol{t})| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed}$$

$$\boldsymbol{r}''(t) = \lim_{h\to 0} \frac{\boldsymbol{r}'(t+h) - \boldsymbol{r}'(t)}{h}$$

$$= \frac{\text{change in velocity}}{\text{change in time}}$$

$$= \text{acceleration} = \boldsymbol{a}(t)$$

Let t be **time in seconds** and assume the position of an object (in **feet**) is given by

$$\boldsymbol{r}(t) = < t, 2e^{-t}, 0 >$$

Compute

1. $\boldsymbol{r}'(t)$, $|\boldsymbol{r}'(t)|$, and $\boldsymbol{r}''(t)$.
2. $\boldsymbol{r}'(0)$, $|\boldsymbol{r}'(0)|$, and $\boldsymbol{r}''(0)$.

HUGE application:
Modeling ANY motion problem.

Newton's 2nd Law of Motion states

Force = mass · acceleration

$$\boldsymbol{F} = m \cdot \boldsymbol{a} \text{ , so}$$

$$\boldsymbol{a} = \frac{1}{m} \cdot \boldsymbol{F}$$

If $\boldsymbol{F} = \langle 0,0,0 \rangle$, then all the forces 'balance out' and the object has no acceleration. (Velocity will remain constant)

If $\boldsymbol{F} \neq \langle 0,0,0 \rangle$, then acceleration will occur, and we integrate (or solve a differential equation) to find velocity and position.

That is how we can model ALL motion problems!

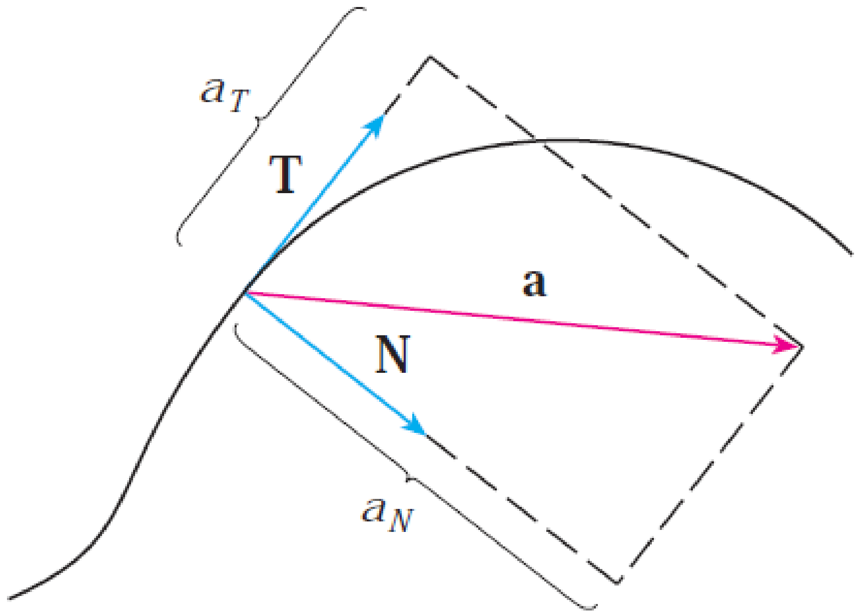
HW Example:

An object of mass 10 kg is being acted on by the force $\boldsymbol{F} = \langle 130t, 10e^t, 10e^{-t} \rangle$. You are given

$\boldsymbol{v}(0) = \langle 0, 0,1 \rangle$ and $\boldsymbol{r}(0) = \langle 0, 1,1 \rangle$.

Find the position function.

Measuring and describing acceleration



Recall: $\text{comp}_{\boldsymbol{b}}(\boldsymbol{a}) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b}}$ = length.

We define the tangential and normal components of acceleration by:

$a_T = \text{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{T}$ = tangential

$a_N = \text{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{N}$ = normal

For computing use,

$$a_T = \frac{\vec{\boldsymbol{r}}' \cdot \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} \quad \text{and} \quad a_T = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|}$$

For interpreting use,

$a_T = \nu' = \frac{d}{dt}|r'(t)|$ = “deriv. of speed”

$a_N = k\nu^2$ = curvature · (speed)^2

Example:

$$\vec{\boldsymbol{r}}(t) = < \cos(t), \sin(t), t >$$

Find the tangential and normal components of acceleration.

Deriving interpretations:

Note that: $\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$

Let $\nu(t) = |\vec{\boldsymbol{v}}(t)|$ = speed.

1. $\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \frac{\vec{\boldsymbol{v}}(t)}{\nu(t)}$ implies $\vec{\boldsymbol{v}} = \nu\vec{\boldsymbol{T}}$.

2. $\kappa(t) = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{T}}'|}{\nu(t)}$ implies $|\vec{\boldsymbol{T}}'| = \kappa\nu$.

3. $\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \frac{\vec{\boldsymbol{T}}'}{\kappa\nu}$ implies $\vec{\boldsymbol{T}}' = \kappa\nu\vec{\boldsymbol{N}}$.

Differentiating the first fact above gives

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + \nu\vec{\boldsymbol{T}}', \text{ so}$$

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{v}}' = \nu'\vec{\boldsymbol{T}} + k\nu^2\vec{\boldsymbol{N}}.$$

Conclusion:

$a_T = \nu' = \frac{d}{dt}|r'(t)| =$ "deriv. of speed"

$a_N = k\nu^2 =$ curvature · (speed)^2